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Sequential updates for non-abelian SOC models

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Abstract. It is well known that the order in which the sites of a non-Abelian coupled map lattice model (as the Olami-Feder-Christensen model) are updated determines the final configuration. In order to eliminate this ambiguity one must use a parallel update. In this paper we present a simple sequential update that is equivalent to the parallel one; we show that it obeys the natural branching structure of the avalanche. We also show that the main effect of the other sequential methods, which do not obey the branching structure of the avalanche, is to increase the revisitations of critical sites, which enhances the number of large avalanches in the lattice.

PACS. 05.45.Ra Coupled map lattices – 05.65.+b Self-organized systems

Since the idea of self-organized criticality (SOC) was proposed [1], a lot of simple models [2–5] have been suggested to describe the threshold dynamics of open extended systems which evolve spontaneously towards a scale-invariant state. In general, these models correspond to cellular automata or coupled maps defined on a lattice, in which an avalanche is triggered when some site variable reaches a threshold value, *id est*, when this site becomes unstable. For the majority of these models (known as non-Abelian [6]), if two sites become unstable (critical) at the same time during an avalanche, the final configuration will depend on the order in which they are updated. To eliminate this ambiguity, the sites must be updated in a way that became known in literature as the parallel (or synchronous) update [7].

In this paper we present some particular sequential updates, called by us pseudo-sequential, that are able to reproduce the configurations obtained by the parallel method, being equivalent to them. We show that the other sequential ways of updating the sites, which do not present this characteristic, increases the number of large avalanches in the statistics of events and that this fact is related to the revisitations of critical sites along an avalanche; we conjecture that this phenomenon, even in the correct parallel update, is an important mechanism for the collective behavior of the model. We illustrate these ideas using the Olami-Feder-Christensen (OFC) model.

The OFC slip-stick model was originally proposed to describe the dynamics of earthquakes. It is a coupled map model, defined on a two-dimensional lattice, whose dynamics is based on the Burridge-Knopoff model [8]. A real variable E_{ij} is associated to each site of the lattice. The system is driven by a global perturbation that slowly increases the energy E_{ij} in all sites of the lattice. This process goes on until the energy of one site reaches a threshold value E_{th}. This site becomes unstable and relaxes according to the rules:

$$\begin{cases} E_{ij} \to 0 \\ E_{\mathrm{nn}} \to E_{\mathrm{nn}} + \alpha E_{ij} \end{cases}, \tag{1}$$

where E_{nn} is the energy of the four nearest-neighbors of the site (i,j), and $\alpha \in [0, 1/4]$ is a parameter that controls the degree of local conservation of energy. If one or more nearest-neighbor sites become unstable as a consequence of the relaxation rule (1), such rule is applied again and this process goes on until all sites of the lattice become stable again ($E_{ij} < E_{\mathrm{th}}\,, \forall\ (i,j)$). When $\alpha = 1/4$, this model is locally conservative. The avalanche size corresponds to the number of times that the relaxation rule (1) is applied since the avalanche has been triggered.

The differences between the sequential and the pseudo-sequential updates can be observed in Figure 1. In the sequential update, after the updating of some critical site, the search to verify the occurrence of another unstable site on the lattice always starts from site $(1,1)$. It continues following always the same order, imposed by the geometry of the lattice, until the next critical site is found, as it is shown in Figure 1a. This site is then immediately updated, and a new search starts, as always, from site $(1,1)$; the search stops when all sites in the lattice are stable (that indicates the end of an avalanche). In the pseudo-sequential update, after the relaxation of some critical site (i,j), the search for other critical sites goes on from this site, starting at the site $(i, j+1)$, as it may be seen in Figure 1b. When some new critical site is found,

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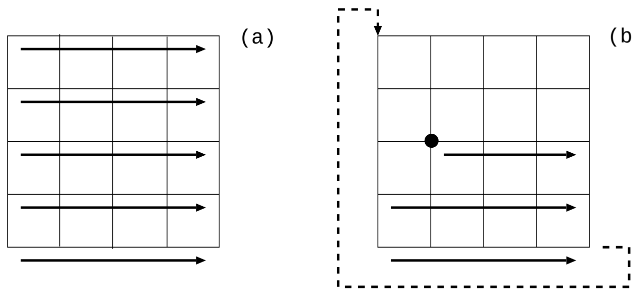

Fig. 1. The search's scheme of critical sites during an avalanche on a two-dimensional lattice if the update is: (a) sequential; (b) pseudo-sequential.

it is immediately updated and the search goes on from that point towards the end of the lattice. Only after that, in order to complete the search, it goes back to site $(1,1)$ until all lattice sites are stable.

In the parallel update, all sites which became critical after the relaxation of sites in the tth generation are updated simultaneously. So the natural evolution of the avalanche establishes a tree of generations, that we will refer, from now on, as the natural hierarchy of generations. We will see that the pseudo-sequential update does not violate the natural tree structure that appears in the parallel update.

In order to illustrate the main differences between these updates, and to show how the pseudo-sequential update obey the natural hierarchy of generations of the avalanche, let us analyze in detail an example of the evolution of an avalanche in the OFC model, according to the three ways of updating the lattice sites (parallel, pseudo-sequential and sequential). In this example we show the intermediate configuration of the avalanching process, after each time step. In all three cases we will begin from the same initial configuration, defined on a small lattice, (3×3), with $E_{\rm th}=4$ and $\alpha=0.25$. The underlined and bold-faced sites correspond to the sites updated at that time step.

Parallel			Pseudo-sequential			Sequential		
3.26	3.35	3.60	3.26	3.35	3.60	3.26	3.35	3.60
2.95	2.75	3.85	2.95	2.75	3.85	2.95	2.75	3.85
3.15	**4.0**	3.75	3.15	**4.0**	3.75	3.15	**4.0**	3.75

↓ ↓ ↓

3.26	3.35	3.60	3.26	3.35	3.60	3.26	3.35	3.60
2.95	3.75	3.85	2.95	3.75	3.85	2.95	3.75	3.85
4.15	0.0	**4.75**	4.15	0.0	**4.75**	**4.15**	0.0	4.75

↓ ↓ ↓

3.26	3.35	3.60	3.26	3.35	3.60	3.26	3.35	3.60
3.99	3.75	**5.04**	2.95	3.75	**5.04***	3.99	3.75	3.85
0.0	2.22	0.0	4.15	1.19	0.0	0.0	1.04	**4.75**

↓ ↓ ↓

3.26	3.35	**4.86**	3.26	3.35	4.86	3.26	3.35	3.60
3.99	**5.01**	0.0	2.95	5.01	0.0	3.99	3.75	**5.04**
0.0	2.22	1.26	**4.15**	1.19	1.26	0.0	2.23	0.0

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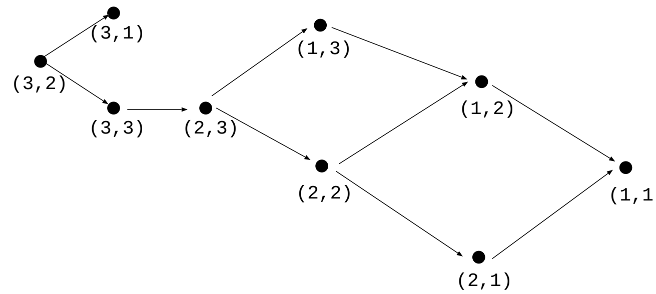

Fig. 2. The generation structure of the avalanche described in the example, considering the usual parallel update of the OFC model. Each branch of the tree corresponds to a generation of the avalanche, in which the critical sites (represented by the points) of the lattice are updated simultaneously. As it may be seen in the example, this avalanche has six generations (it takes six time steps for the complete avalanche evolution).

3.26	**5.82**	0.0	3.26	3.35	**4.86**	3.26	3.35	**4.86**
5.24	0.0	3.85	3.99	5.01	0.0	3.99	5.01	0.0
0.0	3.48	1.26	0.0	2.23	1.26	0.0	2.23	1.26

↓ ↓ ↓

6.02	0.0	1.45	3.26	4.56	0.0	3.26	**4.56**	0.0
0.0	2.76	2.47	3.99	**5.01**	1.21	3.99	5.01	1.21
1.31	3.48	1.26	0.0	2.23	1.26	0.0	2.23	1.26

↓ ↓ ↓

0.0	1.51	1.45	3.26	**5.82**	0.0	**4.40**	0.0	1.41
1.51	2.76	2.47	5.24	0.0	2.47	3.99	6.15	1.21
1.31	3.48	1.26	0.0	3.48	1.26	0.0	2.23	1.26

↓ ↓

4.71	0.0	1.45	0.0	1.10	1.14
5.24	1.45	2.47	**5.09**	6.15	1.21
0.0	3.48	1.26	0.0	2.23	1.26

↓ ↓

6.02	0.0	1.45	1.27	1.10	1.14
0.0	2.76	2.47	0.0	**7.42**	1.21
1.32	3.48	1.26	1.27	2.23	1.26

↓ ↓

0.0	1.51	1.45	1.27	2.96	1.14
1.51	2.76	2.47	1.86	0.0	3.07
1.31	3.48	1.26	1.27	**4.08**	1.26

↓

1.27	2.96	1.14
1.86	1.02	3.07
2.29	0.0	2.28

The generation tree of this avalanche is shown in Figure 2; this avalanche has six generations. The tree can be constructed from the (six) steps of the avalanche according to the parallel update. The points of the tree correspond to critical sites; each branch of the tree defines a new generation of the avalanche. If we also follow the steps of the avalanche according to the other two updates, it is possible to compare their sequences of critical sites with the generations presented in Figure 2 (see Fig. 3a and Fig. 3b).

As shown in Figure 3a, when we consider the pseudo-sequential update, there is no critical site (i,j) of a generation t being updated before another critical site (k,l),

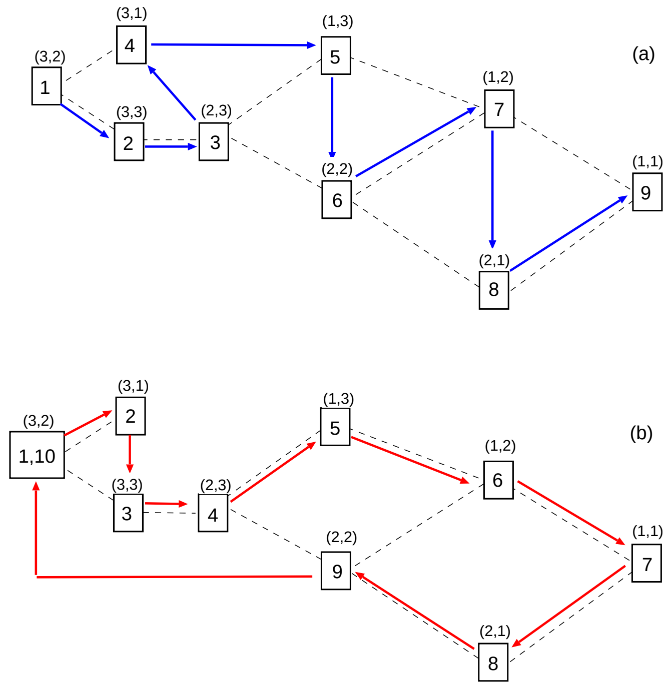

Fig. 3. The sequence in which the critical sites relax, in the avalanche of the example, when the update of the OFC model is: (a) pseudo-sequential; (b) sequential. In both cases, the numeration of the sites corresponds to the order that they are updated and the dashed lines correspond to the generation structure of the avalanche shown in Figure 2.

	$j-2$	$j-1$	j	$j+1$	$j+2$
$i-2$			$\mathbf{c}_N$		
$i-1$		$\mathbf{c}_{NW}$	$\mathbf{b}_N$	$\mathbf{c}_{NE}$	
i	$\mathbf{c}_W$	$\mathbf{b}_W$	$\mathbf{a}$	$\mathbf{b}_E$	$\mathbf{c}_E$
$i+1$		$\mathbf{c}_{SW}$	$\mathbf{b}_S$	$\mathbf{c}_{SE}$	
$i+2$			$\mathbf{c}_S$		

Fig. 4. The first three generations of an avalanche using the parallel update: **a** is the first critical site and corresponds to the generation 0; the sites **b** correspond to the generation 1 and the sites **c**, to the generation 2. We adopt the following notation in order to localize the sites in relation to generation 0 of the avalanche. The subscripts *N, S, E* and *W* indicate north, south, east and west, respectively. Analogously the subscripts *NE, NW, SE* and *SW* indicate northeast, north-west, southeast and south-west, respectively.

originated in a previous generation, and connected to the site (i,j). We say that site (i,j) is connected to site (k,l) if the former received energy from the latter in the parallel update. If a site (i,j) of the generation t is updated before a site (k,l) of the generation $t-1$ that is not connect with it through the generation tree, the final result will not change. For example, the site $(2,3)$ of the generation $t=3$ (which is marked in the example with $*$) is updated before the site $(3,1)$ of the generation $t=2$ but, because there is not a connection between them in the generation tree (see Fig. 2), the final microscopic configuration will not be affected. If this simple rule is followed, the final configuration of the lattice, after the avalanching process, and the avalanche size ($s=9$ in the example) are exactly the same as the parallel update. Note that this does not mean that the intermediate configurations (before the end of the avalanche) are the same as in the parallel upgrade (see the example again).

On the other hand, when we consider the usual sequential update, there are critical sites of a generation t being updated before other critical sites of a generation $t-1$, to which they are connected in the generation tree. For example, we see in Figure 3b that the site $(1,1)$ of the generation $t=6$ is updated before the site $(2,1)$ of the generation $t=5$; and, according to Figure 2, they are connected sites since the site $(1,1)$ received energy from the site $(2,1)$ in the parallel update. The same situation occurs again with the site $(2,1)$ of generation $t=5$ and the site $(2,2)$ of generation $t=4$. In all these cases, the natural hierarchy of generations is violated and, as a consequence, the final configuration of the lattice is different from the configuration generated by the parallel update. The avalanche size is also different. We can observe, in this particular example, that the site $(3,2)$ became critical twice. We say that this site has been revisited along that avalanche. As a consequence, in this example, the avalanche size is $s=10$ for the sequential update.

To show that the above example represents the general case we will use both logical and numerical arguments. First we will present some reasoning showing that no configuration that naturally appears in the pseudo-sequential algorithm violates the natural hierarchy of generations (in which the sites are updated in the parallel algorithm). Then we will give some numerical evidences that support this reasoning.

Suppose that, in a lattice, the site (i,j) (site **a** in Fig. 4) becomes critical, starting an avalanche. Suppose also that the model is conservative ($\alpha=0.25$), and that we have an hypothetical limit situation in which the energy $E_{k,l} \approx E_c$ for a cluster of sites around site **a**, so that, when site **a** relaxes, all of its nearest neighbors (and the nearest neighbors of them) will become critical. Such a configuration maximizes the probability of finding a violation in the hierarchy of generations defined in the parallel update. So, when (i,j), site **a** in Figure 4, relaxes, all of its nearest neighbors, that are sites $(i\pm1, j\pm1)$, become critical. These are the sites labelled $\mathbf{b}_N$, $\mathbf{b}_E$, $\mathbf{b}_S$ and $\mathbf{b}_W$ in Figure 4, and make up the second generation of critical sites in the parallel update. After that, all sites **b** relax simultaneously, and sites $\mathbf{c}_N$, $\mathbf{c}_{NE}$, ... , $\mathbf{c}_{NW}$ plus site **a** again become critical, defining the third generation.

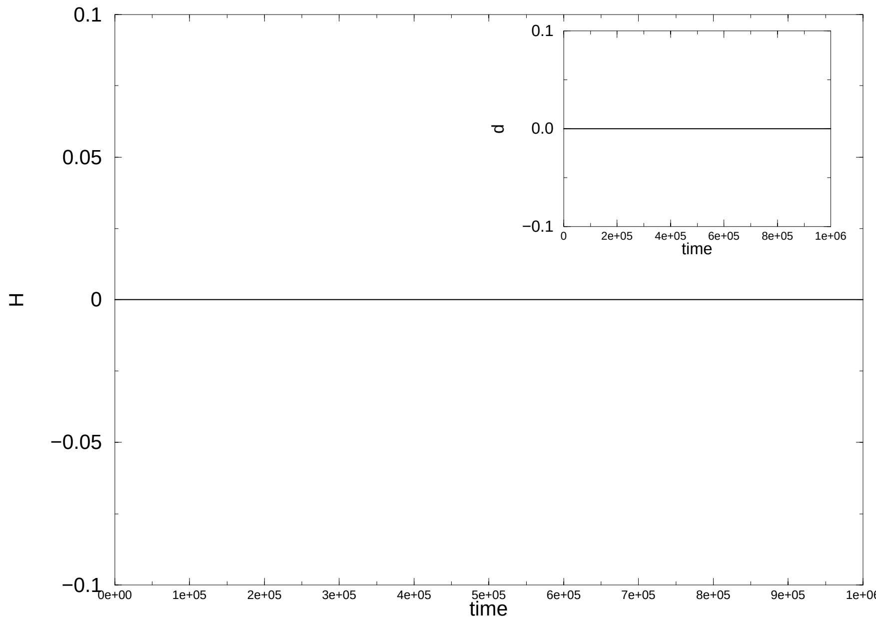

Fig. 5. The Hamming distance between the configuration of the lattice obtained at the end of every avalanche using the pseudo-sequential algorithm (in comparison with the parallel one). The inset shows the difference between the avalanche size using the pseudo-sequential algorithm and the parallel one. For these numerical simulations, the lattice size is $L = 100$ and $\alpha = 0.25$.

We can stop in this third generation, without loss of generality.

Now we can follow this avalanching process according to the sequential and the pseudo-sequential algorithms. In the sequential algorithm, after the relaxation of site $\mathbf{a}$, the next site to relax will be site $\mathbf{b}_N$ (sites $\mathbf{c}_N$, $\mathbf{c}_{NE}$ and $\mathbf{c}_{NW}$ will then become critical), followed by sites $\mathbf{c}_N$ and $\mathbf{c}_{NW}$, both belonging to the third generation. The fact that site $\mathbf{c}_N$ relaxes before sites $\mathbf{b}_E$, $\mathbf{b}_S$ and $\mathbf{b}_W$ is not a problem, because they are not connected in the sense defined before. But note that site $\mathbf{c}_{NW}$ is updated before site $\mathbf{b}_W$, from which it receives some energy in the parallel update. This inversion violates the natural hierarchy of generations and the final configuration of the lattice (the configuration at the end of the avalanche) will be different now, as clearly shown in the example of the small lattice (3×3).

Consider now the same process with the pseudo sequential algorithm. After site $\mathbf{a}$, site $\mathbf{b}_E$ will relax, followed by sites $\mathbf{c}_E$, $\mathbf{b}_S$, $\mathbf{c}_{SE}$, and $\mathbf{c}_S$ (note that site $(i+1, j-1)$, that is, site $\mathbf{c}_{SW}$, has not become critical yet). In this case, when a site of generation $n+1$ is updated, all connected sites of generation n has already been updated. The reader can follow this example further to convince himself of that. It may be observed that, after the toppling of site $\mathbf{c}_S$, the first critical site to be found by the search is not site $\mathbf{c}_N$, since, at this step of the avalanching process, this site has not become critical yet; it only becomes critical after site $\mathbf{b}_N$ relaxes. So, after the toppling of site $\mathbf{c}_S$, when the search starts again from site $(1, 1)$, the first critical site to be found is site $\mathbf{b}_N$. Only after the toppling of site $\mathbf{b}_N$, sites $\mathbf{c}_N$ and $\mathbf{c}_{NW}$ will become critical, but note that, now, they will only be found by the searching algorithm in the next time the search starts at site $(1, 1)$. So, site $\mathbf{c}_N$ will not be updated before site $\mathbf{b}_N$.

This reasoning is still valid if we consider bigger avalanches with more generations. It is also still valid if some of the nearest (or next nearest) neighbors of site $\mathbf{a}$ do not become critical.

Some numerical results of the Hamming distance, to be presented in this section, confirm this fact. We calculated numerically the Hamming distance between the configurations of the lattice at the end of every avalanche, in the parallel and both pseudo-sequential and sequential algorithms. That is, we calculate

$$H \equiv \sum_{i,j} |E_{i,j} - E'_{i,j}|$$

where the sum is over all sites of the lattice, and $E_{i,j}$, $E'_{i,j}$ are the energy of site (i, j) after a parallel and a pseudo-sequential (or sequential) updates, respectively. We also analyzed the difference between the size of each avalanche, $d = s - s'$, during the same sequence of events, where s is the size of an avalanche that evolved according to a parallel update, and s' is the size of the same avalanche now evolving according to a pseudo-sequential (or sequential) update. The simulations were performed with double precision variables, in 100×100 lattices, with runs of 1 000 000 events in the pseudo-sequential case, and of 500 000 events in the sequential case. The results are shown in Figures 5 and 6. We see that both H and d are rigorously zero in the case of the pseudo-sequential algorithm, and clearly different from zero when the sequential update is employed. To make sure this result was not a coincidence, we also repeated the simulations with the pseudo-sequential algorithm, now in smaller runs of 10 000 events, for 500

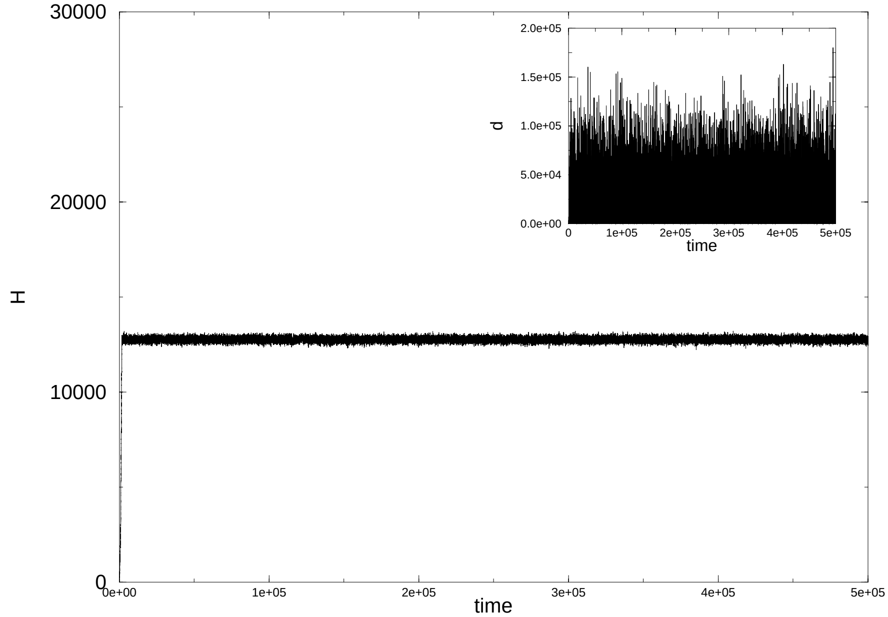

Fig. 6. The Hamming distance between the configuration of the lattice obtained at the end of every avalanche using the sequential algorithm (in comparison with the parallel one). The inset shows the difference between the avalanche size using the sequential algorithm and the parallel one. For these numerical simulations, the lattice size is $L = 100$ and $\alpha = 0.25$.

different initial conditions and observed exactly the same behavior, *i.e.*, both H and d are equal to zero.

After examining the configuration of the lattice at the end of each avalanche, that gives the "microscopical" difference between the sequential and parallel updates, it is interesting to analyze the statistical distribution of the avalanche sizes. It is obvious, as we can see in Figure 7, that there is no difference between the probability distribution of avalanche sizes for the OFC model according to the pseudo-sequential and parallel updates, since the final state of the lattice, after each avalanche, are the same. But another surprising result is that the critical exponent of avalanche sizes, resulting from the sequential update, is the same as the one resulting from the parallel update (see Fig. 7). The different behavior only appears in the cut off region of the distribution, where we may see an increase of the relative number of large avalanches.

Since the number of large avalanches increases when the lattice sites are updated according to a sequential algorithm, the mean value of the avalanche size $\langle s \rangle$ also increases. Comparing this value with the percentage of avalanches with revisited critical sites, we have some evidences that this behavior is related to the increase in the probability of having a critical site revisited. The effect of the update in the mean value of the avalanche size becomes even more evident when the value of parameter α is lower, since, in this case, the occurrence of avalanches with revisited sites is not a predominant effect. For $\alpha = 0.25$, for instance, the relative deviation of the $\langle s \rangle$ between the parallel and sequential update is about 5% in contrast with the results shown in Table 1 for $\alpha = 0.22$.

This comparative analysis, between the parallel and the sequential algorithms clears up the relationship between the revisitation of critical sites and the occurrence of large avalanches without the enlargement of the clusters of sites that take part in the avalanching process. Even in the non-ambiguous parallel update, for high values of parameter α, there is a significant number of large avalanches with revisited critical sites. It is believed that, in the OFC model, the aperiodicity of boundary conditions [10] and the initial disorder [11] are essential elements for the emergence of SOC. This "noise" destroys

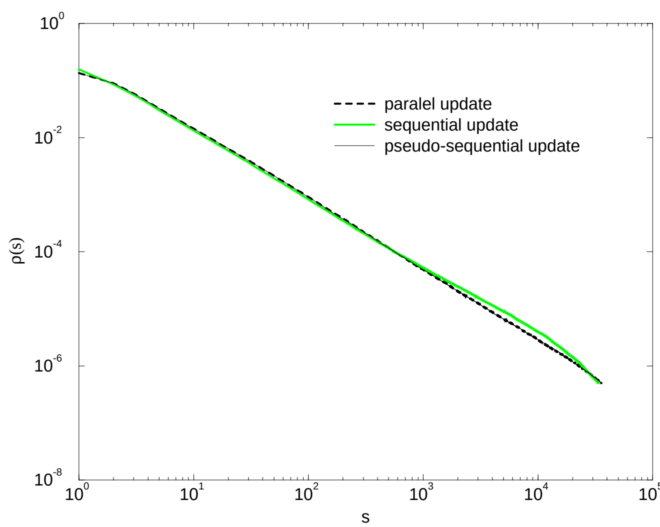

Fig. 7. Probability distribution of the avalanche sizes (log-log graphic) of OFC model in the squared lattice with $L = 100$ and $\alpha = 0.25$ considering the parallel, pseudo-sequential and sequential updates.

Table 1. Analysis of the mean value of avalanche size and the percentage of critical sites revisited for the OFC model considering the parallel and sequential updates; the values of parameter α, the lattice size and the total number of avalanches are, respectively $\alpha = 0.22$, $L = 100$ and $N = 2 \times 10^6$.

	Parallel	Sequential	Relative deviation
$\langle s \rangle$	14.5 ± 0.3	35.9 ± 0.4	147.2%
p_{rev}	1.5%	3.4%	122.1%

the natural tendency of the model to synchronize, allowing the occurrence of large avalanches and the appearance of a power-law behavior. It is also a well known result that the random version of the OFC model [12], whose dynamical rules do not obey a lattice structure, does not display a critical behavior in the non conservative regime. We observe that the revisitation of sites also collaborates to the destruction of the synchronized behavior. This mechanism is not an effect of the boundary conditions, since it may be frequent even for avalanches which do not reach the border of the lattice.

Recently the presence of SOC in the OFC model in the dissipative regime, even with open boundary conditions, has been discussed [13]; it has been suggested that the OFC model has an "almost critical" [9] behavior when $\alpha < 0.25$. If this is true, the above elements, including the revisitation of critical sites, are also an important ingredient for the "almost critical" behavior, inasmuch as this behavior seems also to be induced by the presence of noise.

In conclusion, we showed a simple update (the pseudo-sequential one) of the non-Abelian OFC model that is able to reproduce the results obtained by the usual parallel update. By this we mean that not only the statistics of events (avalanche sizes, for instance) are equal, but that, step by step in time, the system evolves through the same intermediate metastable states (configurations of the lattice after each avalanche). We also observed that the (ambiguous) sequential update, although displaying a power-law behavior with the same critical exponent obtained by the parallel algorithm, enhances the number of large avalanches, that distorts the cutoff region of the distribution. We related this increased number of large avalanches to a significant increase in the number of avalanches with revisited sites during the avalanching process.

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